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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Fotonap program has been modified (i) to include the option to compute the desired solution without inverting the normal equations matrix (thus not obtaining the solution covariance matrix), and (ii) to include the capability to handle Geoceiver measurements. For a typical run involving photogrammetric data, implementation of item (i) reduced the time taken to obtain the normal equations solution from 1 hour 50 minutes to 20 minutes. Two types of Geoceiver measurements have been modeled in Fortnap: satellite-to-ground and

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satellite-to-satellite. Each Geoceiver type may use either a fixed or a variable averaging time. Included in the implementation of the Geoceiver measurements is the Hopfield model for computing tropospheric refraction corrections.

The above modifications have been implemented both on the UNIVAC 1108 and the CDC 6400 versions of Fotonap.

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MODIFICATIONS TO FOTONAP.

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Preface. Authorized funding for implementing the changes to the UNIVAC 1108 and CDC 6400 versions of Fotonap was made by the Defense Mapping Agency. Work was supervised by the United States Army Engineer Topographic Laboratories, Fort Belvoir, Virginia under contract number DAAK 70-77-F-0130. Mr. A. T. Blackburn served as Project Engineer for the government.

1. Introduction

This report describes the mathematical analysis on which the program modifications to Fotonap are based. Implementation of the option not to compute the covariance matrix of the solution vector did not involve any new analysis and therefore is not described here. It did, however, involve the programming of a completely new back-substitution subroutine (INVRTC). The time saving achieved when this option was exercised was also quite significant, the time taken to obtain the solution of the normal equations matrix being reduced from 1 hour 50 minutes to 20 minutes for a typical run using photogrammetric measurements. The main part (Section 2) of this report describes Geoceiver measurements.

Section 3 deals with the Hopfield tropospheric refraction correction formula. Although the formula used in Fotonap is algebraically similar to the NSWC version of the Hopfield model, it is computationally rather different. Also, in the Fotonap version of the Hopfield model, default values for temperature and pressure are adjusted for station height.

The Fotonap User's Guide has been modified to be consistent with the new version of the program. The changes to the User's Guide are described in Section 4.

As the changes to Fotonap to handle Geoceiver measurements had to be fairly extensive, some changes to the program structure for computing predicted measurements (involving subroutines PRTIAL, MEASUR, MESOLD) were made in order to facilitate such changes. In order to check that these changes did not introduce errors in the computation of existing measurement types, a test case involving all

different measurement types was devised. Interestingly, this revealed errors in the computation of some of the partial derivatives of the old version of the program. The analysis for these changes to program are not included in this report, but are based on the analysis in reference 9.

2. Geoceiver Measurements

Although geoceiver measurements may be regarded as measurements of range differences, the correspondence between the two is exact only in the absence of an atmosphere between the transmitter and the receiver. In practice, of course, the atmosphere cannot be removed. However, the atmospheric effect may be removed. The two parts of the atmosphere, the ionosphere and the troposphere, have quite different effects on the measurements and must therefore be treated separately. The ionospheric effect, which is frequency dependent, may be estimated through the use of a two-frequency transmission. The tropospheric effect may be calculated with the aid of a tropospheric model. In order to do so, however, it is necessary to know the geometrical relationships between the transmitter, the receiver and the troposphere. For that reason, tropospheric corrections are usually computed in an orbit determination program rather than in a preprocessor. (Section 3 of this report describes the NWL-Hopfield tropospheric model and how it is used in Photonap to compute tropospheric corrections.) In the following description of geoceiver measurements it will be assumed that atmospheric corrections have been made to the data. For this reason no further mention will be made of the atmosphere.

A satellite transmits a constant frequency signal that is received by the geoceiver. The received frequency will, because of the motion of the transmitter relative to the receiver, differ from the transmission frequency by the (one-way) Doppler frequency.

Let $R(t)$ denote the retarded range of the transmitter relative to receiver at time t. In other words, $R(t)$ is the length of the signal path for a signal being received at time t. Furthermore, let $\dot{R}(t)$ denote the derivative of $R(t)$ with respect to t, and let c denote the transmission velocity. Then clearly

$$\frac{\dot{R}(t)}{c} = \frac{\delta\nu(t)}{\nu_0} \tag{2.1}$$

where ν_0 is the transmission frequency, and $\delta\nu(t)$ is the Doppler frequency at time t. Integrating the above equation between times $(t - T)$ and t, we deduce that

$$g(t) = R(t) - R(t-T), \tag{2.2}$$

where the geoceiver measurement $g(t)$ is given by

$$g(t) = \frac{c}{\nu_0} \int_0^T \delta\nu(\tau + t - T)\, d\tau. \tag{2.3}$$

The geoceiver equipment may be either ground based or carried by a satellite. The first case is the more common one and will be considered first.

2.1 Satellite-to-Ground Measurements

Let $\bar{x}_{50}(t)$ denote the satellite position vector at time t in the 'Mean of date, 1950.0' coordinate system. (This is the inertial coordinate system used by Photonap for all orbit computations.) Also, let R_{D5} denote the rotation matrix transforming vectors in the '1950.0' system to the 'true of date system' at time t. Although R_{D5} obviously is a function of time, it may be regarded as constant over a period of about a minute. The rotation

matrix $R_{FD}(t)$, which transforms an 'of date' vector to an 'Earth fixed' vector is defined by

$$R_{FD}(t) = \begin{bmatrix} \cos A(t) & \sin A(t) & 0 \\ -\sin A(t) & \cos A(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.4}$$

where $A(t)$ is the Greenwich hour angle, and $\dot{A}(t) = \omega$, the Earth's rotation rate.

The inertial satellite vector $\bar{x}(\tau,t)$ is defined by

$$\bar{x}(\tau,t) = R_{FD}(t)\; R_{D5}\, \bar{x}_{50}\,(\tau) \tag{2.5}$$

Equation (2.5) can be seen to represent the satellite vector in an inertial coordinate system that coincides with the 'Earth fixed' coordinate system at time $\tau = t$.

Denoting the 'Earth fixed' station-vector by $\bar{S}_F$, the retarded range, $R(t)$, must, by definition, satisfy the equation

$$R(t) = \sqrt{\left[\bar{S}_F - \bar{x}(\tau,t)\right] \cdot \left[\bar{S}_F - \bar{x}(\tau,t)\right]}, \tag{2.6}$$

where, $$\tau = t - R(t)/c \tag{2.7}$$

The simultaneous equations (2.6) and (2.7) cannot be solved explicitly. However, on account of $R(t)/c$ being small, the equations can very easily be solved using an iterative technique.

The predicted geoceiver measurement is then computed using equation (2.2).

2.1.1 Measurement Partial Derivatives

Let $\bar{q}(t)$ be defined by

$$\bar{q}(t) = \frac{\partial}{\partial p} \bar{x}_{50}(t) \tag{2.8}$$

where p is any parameter affecting the orbit. (p may, e.g., be a component of the initial position vector or a coefficient of one of the spherical harmonics of the gravity field. Photonap computes a vector $\bar{q}$ for each unknown parameter affecting the orbit.)

Squaring both sides of equation (2.6) and differentiating with respect to p we obtain

$$R(t) \frac{\partial}{\partial p} R(t) = \left[\bar{x}(\tau,t) - \bar{S}_F\right] \cdot \left[\frac{\partial}{\partial p} \bar{x}(\tau,t) + \frac{\partial}{\partial \tau} \bar{x}(\tau,t) \frac{\partial \tau}{\partial p}\right] \tag{2.9}$$

Writing

$$\bar{u} = \left[\bar{x}(\tau,t) - \bar{S}_F\right] / R(t) \tag{2.10}$$

and

$$\dot{R}_0 = \bar{u} \cdot \frac{\partial}{\partial \tau} \bar{x}(\tau,t) \tag{2.11}$$

we find with the aid of equation (2.7) that

$$\frac{\partial}{\partial p} R(t) = \bar{u} \cdot \frac{\partial}{\partial p} \bar{x}(\tau,t) + \dot{R}_0 \left[- \frac{1}{c} \frac{\partial}{\partial p} R(t)\right] \tag{2.12}$$

It hence follows from the above and equations (2.5) and (2.8) that

$$\frac{\partial}{\partial p} R(t) = \bar{u}_0 \cdot R_{FD}(t)\, R_{D5} \bar{q}(\tau), \tag{2.13}$$

where

$$\bar{u}_0 = \bar{u}/(1 + \dot{R}_0/c). \tag{2.14}$$

Differentiating the square of equation (2.6) with respect to the station vector $\bar{S}_F$ we similarly obtain

$$R(t)\,\frac{\partial}{\partial \bar{S}_F} R(t) = \left[\bar{x}(\tau,t) - \bar{S}_F\right]\cdot\left[\frac{\partial}{\partial \tau}\,\bar{x}(\tau,t)\,\frac{\partial \tau}{\partial \bar{S}_F}\right] - \left[\bar{x}(\tau,t) - \bar{S}_F\right],$$

whence

$$\frac{\partial}{\partial \bar{S}_F} R(t) = \dot{R}_0\left[-\frac{1}{c}\,\frac{\partial}{\partial \bar{S}_F} R(t)\right] - \bar{u}.$$

We hence deduce that

$$\frac{\partial}{\partial \bar{S}_F} R(t) = -\,\bar{u}_0 \tag{2.15}$$

To obtain the time derivative $\dot{R}(t)$, we similarly deduce from equation (2.6) that

$$R(t)\dot{R}(t) = \left[\bar{x}(\tau,t) - \bar{S}_F\right]\cdot\left[\frac{\partial}{\partial t}\,\bar{x}(\tau,t) + \frac{\partial}{\partial \tau}\,\bar{x}(\tau,t)\,\frac{\partial \tau}{\partial t}\right],$$

whence by equations (2.10), (2.11) and (2.7)

$$\dot{R}(t) = \bar{u}\cdot\frac{\partial}{\partial t}\,\bar{x}(\tau,t) + \dot{R}_0\left[1 - \frac{1}{c}\,\dot{R}(t)\right].$$

From the above equation it follows that

$$\dot{R}(t) = \dot{R}_0/(1 + \dot{R}_0/c) + \bar{u}_0\cdot\frac{\partial}{\partial t}\,\bar{x}(\tau,t) \tag{2.16}$$

It may readily be verified that with the rotation matrix R_{FD} given by equation (2.4)

$$\dot{R}_{FD}\,R_{FD}^{T} = \omega\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.17}$$

Since it follows from equation (2.5) that

$$\frac{\partial}{\partial t}\,\bar{x}(\tau,t) = \dot{R}_{FD}(t)\,R_{DS}\cdot\bar{x}_{S0}(\tau),$$

and $$R_{DS}\,\bar{x}_{50}\,(\tau) = R_{FD}^{T}\,(t)\,\bar{x}(\tau,t),$$

we deduce with the aid of equation (2.17) that

$$\frac{\partial}{\partial t}\,\bar{x}(\tau,t) = \omega\begin{bmatrix} x_2(\tau,t) \\ -x_1(\tau,t) \\ 0 \end{bmatrix} \tag{2.18}$$

Furthermore, since

$$(x_1-S_1)x_2 - (x_2-S_2)x_1 = (x_1-S_1)S_2 - (x_2-S_2)S_1,$$

it follows from equation (2.16) that

$$\dot{R}(t) = \dot{R}_0/(1 + \dot{R}_0/c) + \omega\Big[(u_0)_1\,S_2 - (u_0)_2\,S_1\Big] \tag{2.19}$$

2.2 Satellite-to-Satellite Measurements

In this type of measurement the ground station is replaced by the satellite and the satellite is replaced by another satellite, whose orbit in general can be considered known. In this case, equation (2.6) is replaced by

$$R(t)=\sqrt{\Big[\bar{x}_{50}\,(t) - \bar{u}_{50}\,(\tau)\Big]\cdot\Big[\bar{x}_{50}\,(t) - \bar{u}_{50}\,(\tau)\Big]} \tag{2.20}$$

where $\bar{u}_{50}\,(\tau)$ denotes the position vector of the transmitting satellite, and τ satisfies the equation

$$\tau = t - R(t)/c \tag{2.21}$$

2.2.1 Measurement Partial Derivatives

Similarly to the satellite-to-ground case, we find that

$$R(t)\,\frac{\partial}{\partial p}\,R(t) = \left[\bar{x}_{50}(t) - \bar{u}_{50}(\tau)\right]\cdot\left[\bar{q}(t) + \dot{\bar{u}}_{50}\,\frac{1}{c}\,\frac{\partial R(t)}{\partial p}\right]$$

Hence, writing

$$\bar{v} = \left[\bar{u}_{50}(\tau) - \bar{x}_{50}(t)\right]/R(t), \tag{2.22}$$

and

$$\dot{R}_s = \bar{v}\cdot\dot{\bar{u}}_{50} \tag{2.23}$$

we deduce that

$$\frac{\partial}{\partial p}\,R(t) = -\bar{v}_0\cdot\bar{q}\,(t) \tag{2.24}$$

where

$$\bar{v}_0 = \bar{v}/(1 + \dot{R}_s/c) \tag{2.25}$$

Corresponding to equation (2.8), we define

$$\bar{q}_s\,(t) = \frac{\partial}{\partial p_s}\,\bar{u}_{50}(t). \tag{2.26}$$

It then follows that

$$\frac{\partial}{\partial p_s}\,R(t) = -\bar{v}\cdot\left[-\bar{q}_s(\tau) - \dot{\bar{u}}_{50}(\tau)\left(-\frac{1}{c}\,\frac{\partial}{\partial p_s}\,R(t)\right)\right],$$

i.e.,

$$\frac{\partial}{\partial p_s}\,R(t) = \bar{v}\cdot\bar{q}_s(\tau) - \dot{R}_s\left(\frac{1}{c}\,\frac{\partial}{\partial p_s}\,R(t)\right).$$

Hence,

$$\frac{\partial}{\partial p_s}\,R(t) = \bar{v}_0\cdot\bar{q}_s(\tau) \tag{2.27}$$

[Note that if both satellites are functions of the same parameter, such as a gravity coefficient, so that p and p_s are the same parameter, then the partial derivative is simply the sum of the right

hand sides of equations (2.24) and (2.27)]

Differentiating equation (2.20), we obtain for the time derivative

$$\dot{R}(t) = - \bar{v} \cdot \left[\dot{\bar{x}}_{50} - \dot{\bar{u}}_{50}(\tau)(1 - \frac{1}{c}\dot{R}(t))\right],$$

i.e.,

$$\dot{R}(t) = - \bar{v} \cdot \dot{\bar{x}}_{50} + \dot{R}_s\left[1 - \frac{1}{c}\dot{R}(t)\right].$$

Consequently,

$$\dot{R}(t) = - \bar{v}_0 \cdot \dot{\bar{x}}_{50} + \dot{R}_s/(1 + \dot{R}_s/c). \tag{2.28}$$

3.0 The NWL-Hopfield Model for Tropospheric Refraction of Radio Ranging Signals

3.1 The Index of Refraction

The NWL-Hopfield model, like many similar models, is based on the Smith-Weintraub formula (reference 1)

$$N = \frac{77.6}{T}\left[p + 4810\,\frac{e}{T}\right], \tag{3.1}$$

where

N is the refractivity $\{N = (n-1)10^6$, where n is the refractive index$\}$,

T is the temperature (degrees Kelvin),

p is the total pressure (millibars),

e is the partial pressure of water vapor (millibars).

Smith and Weintraub claim that for frequencies below 30,000 MHz the formula is accurate to within 0.5% for temperatures between -50°C and 40°C, pressures between 200 mb and 1100 mb, and partial water vapor pressures between 0 mb and 30 mb.

It is interesting to note that the formula is derived from the three-term expression

$$N = k_1\,\frac{p}{T} + k_2\,\frac{e}{T} + k_3\,\frac{e}{T^2}, \tag{3.2}$$

where, "the first term expresses the sums of the distortions of electronic charges of the dry gas molecules under the influence of an applied electromagnetic field, the second term the distortion for water vapor, and the third term the effect of the orientation of the dielectric dipoles of water vapor under the influence of a field." Consistant with equation (3.1) $k_1 = 77.6$.

k_2 and k_3 were determined by Birnbaum and Chatterjee (reference 2), who obtained $k_2 = 72.0$, $k_3 = 3.75 \times 10^5$. Since the total pressure $p = p_d + e$, equation (3.2) may be rewritten in the form

$$N = k_1 \frac{p}{T} + k_3' \frac{e}{T^2},$$

where $k_3' = k_3 - (k_1 - k_2)T$. Since k_1 and k_2 are nearly equal, the temperature variation of k_3' is so small that for the region of applicability of the formula this may be neglected. Formula (3.1) thus results.

3.2 Pressure and Temperature Variation with Height

In this section the derivation of the required formulae will closely follow that given by Hopfield (references 3, 4, 5). Theoretically, α_T, the rate at which the temperature of dry air decreases with height (the dry adiabatic lapse rate) is given by

$$\alpha_T = \frac{2}{7} \frac{gm}{R}, \tag{3.3}$$

where $g = 981$ cm/sec^2, is the acceleration due to gravity
$m = 29.0$ grams, is the mass of a mole of dry air, and
$R = 8.31 \times 10^7$ ergs/(mole)(deg Celsius) is the Universal Gas Constant.

The theoretical rate, 9.8°C/km, is rather higher than the observed rate, which is closer to 6°C/km. The observed rate does, however, fluctuate. Hopfield realized that if she could choose a lapse rate α, which is an integral fraction of (gm/R), then her tropospheric model would be much simplified. The Hopfield value, 6.84° C/km, corresponds to

$$\alpha = \frac{1}{5} \frac{gm}{R}. \tag{3.4}$$

The temperature lapse rate is assumed to be constant so that

$$T = T_o - \alpha h \tag{3.5}$$

where T_o is the surface temperature and h the height above the surface. Since, clearly the temperature cannot drop below absolute zero, the Hopfield troposphere must possess an upper limit, h_1, given by

$$h_1 = T_o/\alpha \tag{3.6}$$

Assuming the earth to be flat and the acceleration due to gravity to be constant, we deduce that

$$\frac{dp}{dh} = - \rho g, \tag{3.7}$$

where ρ is the density of air. Writing the gas equation PV = RT in the form

$$\rho = \frac{pm}{RT}, \tag{3.8}$$

we obtain with the aid of equation (3.5)

$$\frac{1}{p}\frac{dp}{dh} = - gm/R(T_o - \alpha h),$$

whence,

$$p = p_o(1 - h\,\alpha/T_o)^{\frac{gm}{R\alpha}} \tag{3.9}$$

From the above and equations (3.4), (3.5) and (3.6) we obtain

$$p = p_o(1 - h/h_1)^5 \tag{3.10}$$

and

$$T = T_o(1 - h/h_1) \tag{3.11}$$

Since $0^{o}C$ corresponds to $273.16^{o}K$, it follows that equation (3.6) may be written as

$$h_1 = (273.16 + T_c)/\alpha,$$

where T_c is the surface temperature in degrees Celsius. Using the value of α from equation (3.4) we would obtain

$$h_1 = (39.3 + 0.146\ T_c)\text{km}.$$

The actual formula employed by the Hopfield model is

$$h_1 = (40.1 + 0.149\ T_c)\text{km}. \tag{3.12}$$

3.3 The Zenith Integral for Dry Air

The zenith integral is defined by

$$I_z = \int_0^{h_1} N\,dh, \tag{3.13}$$

where h_1 is the upper limit of the modeled atmosphere. Since $e = 0$ for dry air it follows from equations (3.1), (3.10) and (3.11) that

$$I_z = \frac{77.6\ p_o}{T_o} \int_0^{h_1} (1-h/h_1)^4 dh,$$

i.e.,

$$I_z = \frac{77.6\ p_o h_1}{5\ T_o} \tag{3.14}$$

It is interesting to note that if h_1 were given by equations (3.6) and (3.4), equation (3.14) reduces to

$$I_z = 77.6\ p_o R/gm, \tag{3.14'}$$

so that the zenith integral is a function solely of the surface

pressure. Assuming that the Earth is spherical and that the inverse square law of gravitation applies, this result may also be deduced as follows. If r denotes the radial distance from the Earth's center, then for static equilibrium the net surface force on a small element of volume must balance the body force. Hence,

$$\frac{d}{dr}\left(p\,\frac{r^2}{r_o{}^2}\right) = -\left(\rho\frac{r^2}{r_o{}^2}\right)\left(g_o\frac{r_o{}^2}{r^2}\right),$$

where g_o is the acceleration due to gravity at distance r_o from the Earth's center. Integrating the above equation between r_o and $\bar{r}$, the upper limit of the Earth's atmosphere, we deduce that

$$\int_{r_o}^{\bar{r}} \rho dr = p_o/g_o .$$

It follows from equations (3.1) and (3.8) that the 'dry' refractivity is given by $N = 77.6\ \rho R/m$, whence the zenith integral

$$I_z = 77.6\ p_o R/g_o m. \qquad (3.14")$$

It can thus be seen that the result is fairly general and is not a function of the temperature distribution within the atmosphere.

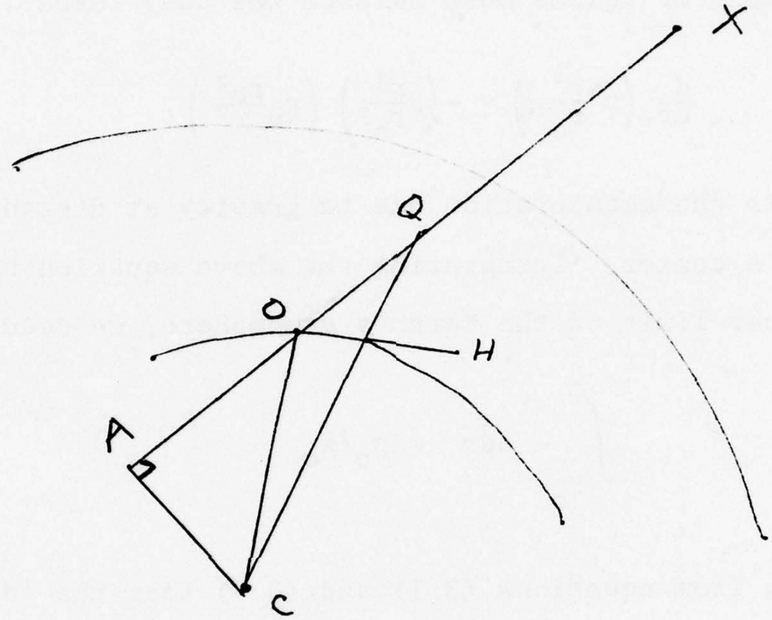
However, h_1 in the Hopfield model is given by equation (3.12) rather than by equation (3.6). The corresponding zenith integral I_z is given by

$$I_z = 2.28 p_o \frac{1 + T_c/269}{1 + T_c/273}\ \text{km}, \qquad (3.15)$$

thus, showing that in the Hopfield model the zenith integral increases with temperature.

FIGURE 3.1

The Transmitter to Receiver Straight Line Path



O is the location of the receiving station

C is the center of curvature at the point O,

X is the location of the transmitter,

OC = $r_\circ$, the radius of curvature

QC = r

OQ = s, the path length

angle QOH = $E_\circ$, the elevation angle at the station

angle OAC = $\frac{1}{2}\pi$.

3.4 The Range Correction

The computed range correction is given by

$$\Delta s = \int_{r_o}^{r_o+h_1} (n-1) \frac{ds}{dr} dr, \tag{3.16}$$

where r_o is the Earth's radius of curvature at the receiving station and ds is an element of path length. The error in performing the integration along the geometric straight line rather than along the actual path is small (reference 3) "reaching 10% of Δs at the horizon but becoming neglibly small at elevation angles above 3° or 4°."

Referring to Figure 3.1, it can easily be seen that $OA = r_o \sin E_o$ and $CA = r_o \cos E_o$. Since by Pythagoras $QC^2 = AQ^2 + CA^2$ it follows that

$$r^2 = (s + r_o \sin E_o)^2 + (r_o \cos E_o)^2.$$

Hence

$$r = (s + r_o \sin E_o)\frac{ds}{dr},$$

and

$$\frac{ds}{dr} = \frac{r}{\sqrt{r^2 - (r_o \cos E_o)^2}} \tag{3.17}$$

Since $N = (n-1)10^6$, it follows from equations (3.1), (3.10), (3.11), (3.16) and (3.17) that the computed 'dry' correction Δs_1, is given by

$$\Delta s_1 = 77.6 \times 10^{-6} \frac{p_o}{T_o} \int_{r_o}^{r_o+h_1} \frac{(1-h/h_1)^4 \, r \, dr}{\sqrt{r^2 - (r_o \cos E_o)^2}}, \tag{3.18}$$

where

$$h = r - r_o . \tag{3.19}$$

Writing,

$$h_1 = r_1 - r_o, \tag{3.20}$$

equation (3.18) may be re-written as

$$\Delta s_1 = 77.6 \times 10^{-6} \frac{p_o}{T_o} \int_{r_o}^{r_1} \left(\frac{r_1 - r}{r_1 - r_o}\right)^4 \frac{r\,dr}{\sqrt{r^2 - (r_o \cos E_o)^2}} \tag{3.21}$$

As pointed out by Yionoulis (reference 6), equation (3.21) may be integrated in closed form, but the numerical solution, even in double precision, is inaccurate at high elevation angles. Yionoulis' answer to the problem was to produce two separate solutions, one being valid for all elevation angles except the low ones, the other for all angles except the high ones. A simpler solution is obtained by using the NWL approximation (Reference 7). This will now be described.

Let $$N_1 = 77.6\ p_o/T_o\ 10^6 \tag{3.22}$$

Since,

$$\frac{r_1 - r}{r_1 - r_0} = \left(\frac{r_1^2 - r^2}{r_1^2 - r_0^2}\right)\left(\frac{r_1 + r_0}{r_1 + r}\right) \tag{3.23}$$

and since $(r_1 + r_0)/r_1 + r)$ is approximately equal to one, it follows that Δs_1 is approximately equal to

$$\Delta s_1 = N_1 \int_{r_0}^{r_1} \left(\frac{r_1^2 - r^2}{r_1^2 - r_0^2}\right)^4 \frac{r\,dr}{\sqrt{r^2 - r_0^2 \cos^2 E_0}} \tag{3.24}$$

A more rigorous justification for the NWL approximation (3.24) is given in Appendix B.

Making the substitutions

$$\begin{aligned} x &= r^2 - r_0^2 \cos^2 E_0 \\ x_1 &= r_1^2 - r_0^2 \cos^2 E_0 \\ x_0 &= r_0^2 - r_0^2 \cos^2 E_0 \end{aligned} \tag{3.25}$$

we deduce that

$$\Delta s_1 = \tfrac{1}{2}N_1 \int_{x_0}^{x_1} \left(\frac{x_1-x}{x_1-x_0}\right)^4 \frac{dx}{\sqrt{x}} \tag{3.26}$$

A closed form solution for the above integral (reference 7) is easily obtained. However, if Δs_1 is computed by the usual method of evaluating the solution at both limits, then the result will be numerically inaccurate, particularly at high elevation angles. A better way to evaluate the integral is given in Appendix A. It follows from equations (3.26) and (3.25), and equations (1), (3), (12) and (13) in Appendix A that,

$$\Delta s_1 = \frac{1}{5} N_1 I_0 (1 + \frac{2}{3}t + \frac{2}{7}t^2 + \frac{1}{14}t^3 + \frac{1}{126}t^4), \tag{3.27}$$

where

$$\left.\begin{aligned} I_0 &= (r_1-r_0)(r_1+r_0)/d \\ d &= \sqrt{(r_1-r_0)(r_1+r_0) + r_0^2 \sin^2 E_0} + r_0 \sin E_0 \\ t &= I_0/d \end{aligned}\right\} \tag{3.28}$$

since $d^2 \geq (r_1-r_0)(r_1+r_0)$, and $r_1-r_0 \geq 0$,

if follows that

$$0 < t \leq 1 \tag{3.29}$$

Since the last term on the right hand side of equation (3.27) contributes less than 0.5% to the total integral, it may be neglected. We may therefore write Δs_1 in the truncated form

$$\Delta s_1 = \frac{1}{5} N_1 I_0 \left(1 + \frac{2}{3}t + \frac{2}{7}t^2 + \frac{1}{14}t^3\right) \tag{3.30}$$

3.5 Adjustment for Station Height Above Sea-Level

Let h_0 denote the station height above sea-level, and p_s and T_s the pressure and temperature at sea-level. In accordance with equations (3.10) and (3.11) we must then have

$$p_0 = p_s (1 - h_0/h_1)^5 \tag{3.31}$$

$$T_0 = T_s (1 - h_0/h_1). \tag{3.32}$$

Since by equation (3.12)

$$h_1 = \left[40.1 + 0.149 \ (T_s - 273)\right] \text{km}$$

it follows that

$$h_1 - h_0 = \left[(40.1 - 0.149 \times 273)(1 - h_0/h_1) + 0.149 \ T_c\right] \text{km}$$

i.e.,

$$h_1 - h_0 = \left[40.1 + 0.6 \ h_0/h_1 + 0.149 \ T_c\right] \text{km} \tag{3.33}$$

Since h_1 is of the order of 40km, the ratio h_0/h_1 is small and may be ignored. Following O'Toole (reference 7) we thus obtain

$$h_1 - h_0 = \left[40.1 + 0.149 \ T_c\right] \text{km}, \tag{3.34}$$

In other words, the height of the atmosphere above the station is independent of the station height above sea-level.

Taking station height into account, equation (3.21) must be modified to

$$\Delta s_1 = 77.6 \times 10^{-6} \frac{p_s}{T_s} \int_{r_0+h_0}^{r_1} \left(\frac{r_1 - r}{r_1 - r_0}\right)^4 \frac{r \, dr}{\sqrt{r^2 - (r_0+h_0)^2 \cos^2 E_0}}$$

whence, by equations (3.31) and (3.32)

$$\Delta s_1 = 77.6 \times 10^{-6}\,\frac{p_0}{T_0}\int_{r_0+h_0}^{r_1}\left(\frac{r_1-r}{r_1-r_0}\right)^4 \frac{h_1^4}{(h_1-h_0)^4}\,\frac{r\,dr}{\sqrt{r^2-(r_0+h_0)^2\cos^2E_0}} \tag{3.35}$$

Let $s_0 = r_0 + h_0$, and $k_1 = h_1 - h_0$ (3.36)

Since, by equation (3.20), $h_1 = r_1 - r_0$, it follows that

$$h_1 - h_0 = r_1 - s_0. \tag{3.37}$$

We hence deduce that

$$\Delta s_1 = 77.6 \times 10^{-6}\,\frac{p_0}{T_0}\int_{s_0}^{r_1}\left(\frac{r_1-r}{r_1-s_0}\right)^4 \frac{r\,dr}{\sqrt{r^2-(s_0\cos E_0)^2}}. \tag{3.38}$$

Comparing equations (3.34), (3.37) and (3.38) with equations (3.12), (3.20) and (3.21) we see that the only difference between the equations is that the radius of curvature r_0 has been replaced by the new radius of curvature s_0. Since, furthermore, the range correction is comparatively insensitive to variations in the radius of curvature, we conclude that as long as the pressure and temperature are obtained at the station, the station height above sea-level may be ignored.

3.6 The 'Wet' Correction

The partial pressure of water vapor appearing on the right hand side of equation (3.1) is computed as a function of the fractional relative humidity h_r ($h_r = 1$ corresponding to 100 percent relative humidity) and the saturation water vapor pressure e_s. It is given by the formula

$$e = h_r\, e_s, \tag{3.39}$$

where

$$e_s = \exp\left(1.80910 + \frac{17.269425T_c}{237.3 + T_c}\right) . \qquad (3.40)$$

Equation (3.40) comes from the National Climatic Center in Asheville, N.C., (Hopfield, Private Communication, 1976).

It was observed by Hopfield (reference 4) that "at heights of 9 or 10km, the atmospheric pressure is still almost one third of its surface value, but the partial pressure of water vapor is nearly zero".

Seemingly without any theoretical justification, Hopfield then went ahead to give the 'wet' correction in a form similar to the 'dry' correction, equation (3.12) being replaced by

$$h_2 = 12.0 \text{ km}, \qquad (3.41)$$

and equation (3.22) by

$$N_2 = .373\ e/T_0^2 , \qquad (3.42)$$

the remaining equations being left the same except for the subscript 1 being replaced by 2.

In the original Hopfield model, the 'wet' atmosphere was assumed to have a ceiling of 12km above sea level. In the O'Toole version the ceiling is at 12km above the station! The problem is really that it is extremely difficult to modify a model whose coefficients do not have a clear physical meaning. In photonap the ceiling is also taken at 12km above the station.

3.7 <u>The Radius of Curvature</u>

It is well known (see, e.g., reference 8) that the radius of curvature, R_A, at a point on the Earth's surface, must satisfy

$$\frac{1}{R_A} = \frac{1}{R_N}\cos^2 A + \frac{1}{R_E}\sin^2 A, \tag{3.43}$$

where

$R_N = a(1-e^2)(1-e^2\sin^2 L)^{-3/2}$,

$R_E = a(1-e^2\sin^2 L)^{-\frac{1}{2}}$,

$a = 6378km$ is the semi-major axis

$e = 0.081813$ is the eccentricity

L is the latitude, and

A is the azimuth.

It can thus be seen that the radius of curvature is a maximum (denoted by R_{MAX}) at the pole and a minimum (denoted by R_{MIN}) in the North-South direction at the Equator. We find that

$$R_{MAX} = a(1-e^2)^{-\frac{1}{2}} = 6400km \tag{3.44}$$

$$R_{MIN} = a(1-e^2) = 6335km \tag{3.45}$$

The average radius $\bar{R}$ is given by

$$\bar{R} = a(1-e^2)^{\frac{1}{6}} = 6371km \tag{3.46}$$

In considering the Earth's curvature we observe that (i) it has no effect on ray paths through zenith, and (ii) the proportional effect increases as the elevation decreases. The maximum effect can therefore be seen to occur at zero elevation angles. It can be seen from equation (3.28) that for $E_0 = 0$,

$$I_0 = \sqrt{r_1^2 - r_0^2} \quad \text{and } t = 1,$$

whence we deduce from equation (3.27) and equation (19) in Appendix A that the correction

$$\Delta s_1 = N_1 \sqrt{r_1^2 - r_0^2}\, \frac{128}{315} \tag{3.47}$$

Since $r_1 - r_0 = h_1$ it follows that

$$\Delta s_1(r_0) = \frac{128}{315} N_1 \sqrt{h_1(2r_0 + h_1)}, \tag{3.48}$$

where we have written the left hand side in a form that indicates that it is a function of r_0. Differentiating equation (3.48) logarithmically we obtain

$$\frac{1}{\Delta s_1(r_0)} \frac{d}{dr_0} \Delta s_1(r_0) = \frac{1}{2r_0 + h_1} \tag{3.49}$$

From the above we deduce that approximate relationship

$$\frac{\delta[\Delta s_1(r_0)]}{\Delta s_1(r_0)} = \frac{\delta r_0}{2r_0 + h_1} \tag{3.50}$$

choosing r_0 to be the average radius $\bar{R}$ given by equation (3.46), we see from equations (3.44) and (3.45) that δr_0 must be numerically less than 36km for any point on the Earth's surface. We hence deduce that if we assume that the radius of curvature is a constant then the proportional error (the left hand side of equation (3.50)) is less than .003. In other words, taking

$$r_0 = 6371\text{km} \tag{3.51}$$

leads to an error that does not exceed 0.3 percent.

3.8 Summary

The following equations are required for the computation of the range correction.

$$N_1 = 77.6 \times 10^{-6}\ p_0/T_0, \tag{3.22}$$

where p_0 is the total atmospheric pressure in millibars, and T_0 is the temperature (degrees Kelvin)

$$N_2 = .373\ e/T_0^2, \tag{3.42}$$

$$e = h_r \exp\left(1.80910 + \frac{17.269425T_c}{237.3 + T_c}\right) \tag{3.39}$$

where h_r is the fractional relative humidity, and T_c $(=T_0-273.16)$ is the temperature (degrees Celsius)

$$h_1 = 40.1 + 0.149\ T_c \tag{3.12}$$

$$h_2 = 12.0 \tag{3.41}$$

$\Delta s = \Delta s_1 + \Delta s_2$, is the total correction, where

$$\Delta s_i = N_i d_i t_i \left(\frac{1}{5} + \frac{2}{15}t_i + \frac{2}{35}t_i^2 + \frac{1}{70}t_i^3\right),\ \text{for } i = 1,2, \tag{3.30}$$

where,

$$\left.\begin{aligned} d_i &= \sqrt{h_i(2r_0+h_i) + (r_0\sin E_0)^2} + (r_0\sin E_0) \\ t_i &= h_i(2r_0+h_i)/d_i^2 \end{aligned}\right\} \quad \begin{matrix}(3.28)\\(3.20)\end{matrix}$$

$$r_0 = 6371, \tag{3.51}$$

and E_0 is the elevation angle of the transmitter as seen from the station.

Range corrections, computed for different elevation angles and temperatures, are given in Table 3.1.

TABLE 3.1

TROPOSPHERIC RANGE CORRECTIONS (HOPFIELD MODEL)

Elevation Angle	'Dry' correction (meters) -60°C	-30°C	0°C	30°C	40°C	'Wet' correction (meters) -30°C	0°C	30°C	40°C
0°	94.2	88.3	83.4	79.2	78.0	0.5	4.8	27.3	44.4
1°	63.9	61.3	59.0	57.0	56.4	0.3	2.7	15.1	24.6
2°	46.6	45.4	44.2	43.2	42.9	0.2	1.8	9.9	16.1
3°	36.0	35.3	34.7	34.2	34.0	0.1	1.3	7.2	11.7
4°	29.0	28.7	28.3	28.0	27.9	0.1	1.0	5.6	9.1
6°	20.6	20.5	20.4	20.3	20.2	0.1	0.7	3.8	6.3
8°	15.9	15.9	15.8	15.8	15.8	0.1	0.5	2.9	4.8
10°	12.9	12.9	12.9	12.9	12.9	0.0	0.4	2.4	3.8
15°	8.8	8.8	8.8	8.8	8.8	0.0	0.3	1.6	2.6
20°	6.7	6.7	6.7	6.7	6.7	0.0	0.2	1.2	2.0
30°	4.6	4.6	4.6	4.6	4.6	0.0	0.1	0.8	1.3
40°	3.6	3.6	3.6	3.6	3.6	0.0	0.1	0.6	1.0
60°	2.7	2.7	2.7	2.7	2.7	0.0	0.1	0.5	0.8
90°	2.3	2.3	2.3	2.3	2.3	0.0	0.1	0.4	0.7

The 'dry' correction (Δs_1) has been computed for a pressure of 1013mb. For other pressures the corrections should be adjusted proportionately. The 'wet' correction (Δs_2) applies for 100 percent relative humidity. The total correction (Δs) is given by

$$\Delta s = \Delta s_1 + h_r \Delta s_2$$

where h_r is the fractional relative humidity.

4.0 Changes to the Photonap User's Guide

The most important changes to the User's Guide stem from the added capability to process geoceiver measurements. Geoceiver measurements have been assigned measurement type number 27 (see Appendix IV). Photonap has the capability to process both 'satellite-to-ground' and 'satellite-to-satellite' geoceiver data, the User having to specify the type, which is being processed (see category 701 card, key 8). Photonap computes 'satellite-to-ground' tropospheric corrections using the NWL-Hopfield model. The User may be content to use default values for meteorological data or he may input his own (see category 610 card. The category 610 card is also used to specify the geoceiver time interval). The geoceiver time interval may also be specified as part of the observation record (see Appendix I). Geoceiver data (see Appendix IV) may be used to recover orbital parameters, station locations, refraction parameters, measurement bias, measurement timing bias, measurement drift rate and measurement scale.

Some relatively minor changes from NAP 3.1F, which give Photonap a Monte Carlo capability, have been added to Photonap (for more details, see Appendix B of the 'Use of the GPS Satellite System for the Determination of the MAGSAT Position' NAS5-23587, Georg Morduch, Old Dominion Systems, September 1976). Affected cards are 601 cards (Note 9) and 604/605 cards.

The User may, in the current version of Photonap, request that the covariance of the solution not be computed (see Category 105, set 2).

Description of the mode 2 discrete thrust has been clarified (see Category 208 card).

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APPENDIX A

EVALUATION OF THE INTEGRAL I_n

The integral I_n is defined by

$$I_n = \frac{1}{2}\int_{x_o}^{x_i} \left(\frac{x - x_i}{x_o - x_i}\right)^n \frac{dx}{\sqrt{x}} \tag{1}$$

Let

$$y = \frac{\sqrt{x} - \sqrt{x_o}}{\sqrt{x_i} - \sqrt{x_o}} \tag{2}$$

and

$$t = \frac{\sqrt{x_i} - \sqrt{x_o}}{\sqrt{x_i} + \sqrt{x_o}} \tag{3}$$

Differentiating equation (2) we obtain

$$\frac{1}{2}\frac{dx}{dy} = \sqrt{x}\,(\sqrt{x_i} - \sqrt{x_o}) \tag{4}$$

From equation (3) we deduce that

$$(1 + t)/t = \frac{2\sqrt{x_i}}{\sqrt{x_i} - \sqrt{x_o}} \tag{5}$$

It hence follows from equations (2) and (3) that

$$\left[y - 1\right]\left[y - 1 + (1+t)/t\right] = \frac{x - x_i}{(\sqrt{x_i} - \sqrt{x_o})^2}\ , \tag{6}$$

whence,

$$\frac{x - x_i}{x_i - x_o} = \left[y - 1\right]\left[1 + yt\right] \tag{7}$$

Changing the variable of integration from x to y we obtain with the aid of equations (2), (4) and (7),

$$I_n = \left(\sqrt{x}_i - \sqrt{x}_o\right) \int_0^1 (1-y)^n (1+yt)^n dy \tag{8}$$

Since

$$(1+yt)^n = \sum_{s=0}^{n} \frac{n!}{s!(n-s)!} y^s t^s \tag{9}$$

and

$$\int_0^1 (1-y)^n y^s dy = \frac{n!s!}{(n+s+1)!} \tag{10}$$

it follows that

$$I_n = (\sqrt{x}_i - \sqrt{x}_o) \sum_{s=0}^{n} \frac{(n!)^2}{(n-s)!(n+s+1)!} t^s \tag{11}$$

[Note that the definite integral of equation (10) is the well known Beta function and is denoted by $B(n+1,s+1)$]

Evaluating equation (11) for n = 0, 4 and 5, we find that

$$I_o = \sqrt{x}_i - \sqrt{x}_o \tag{12}$$

$$I_4 = \frac{1}{5}I_o\left(1 + \frac{2}{3}t + \frac{2}{7}t^2 + \frac{1}{14}t^3 + \frac{1}{126}t^4\right) \tag{13}$$

$$I_5 = \frac{1}{6}I_o\left(1 + \frac{5}{7}t + \frac{5}{14}t^2 + \frac{5}{42}t^3 + \frac{1}{42}t^4 + \frac{1}{462}t^5\right) \tag{14}$$

For the special case t = 1, equation (8) becomes

$$I_n = \sqrt{x}_i \int_0^1 (1-y)^n (1+y)^n dy \tag{15}$$

Since (1-y)(1+y) is an even valued function, we may also write equation (15) in the form

$$I_n = \frac{1}{2} \sqrt{x}_i \int_{-1}^1 (1-y)^n (1+y)^n dy \tag{16}$$

Making the substitution

$$u = \frac{1}{2}(1+y)$$

we obtain

$$I_n = 2^{2n} \sqrt{x}_i \int_0^1 u^n(1-u)^n \, du \tag{17}$$

The integral on the right hand side of equation (17) is the form of equation (10). We therefore conclude that if <u>t = 1</u>,

$$I_n = \sqrt{x}_i \quad 2^{2n}(n!)^2/(2n+1)! \tag{18}$$

In particular

$$I_4 = \frac{128}{315} \sqrt{x}_i \tag{19}$$

APPENDIX B

THE NWL APPROXIMATION TO THE HOPFIELD INTEGRAL

In this appendix it will be shown that equation (3.24) is a valid approximation of equation (3.21). Denoting the difference by D, we find that

$$D = N_1 \int_{r_0}^{r_1} \left[\left(\frac{r_1^2 - r^2}{r_1^2 - r_0^2} \right)^4 - \left(\frac{r_1 - r}{r_1 - r_0} \right)^4 \right] \frac{r\,dr}{\sqrt{r^2 - r_0^2 \cos^2 E_0}} \qquad (1)$$

$$= N_1 \int_{r_0}^{r_1} \left(\frac{r_1^2 - r^2}{r_1^2 - r_0^2} \right)^4 \frac{r\,dr}{\sqrt{r^2 - r_0^2 \cos^2 E_0}} \; G, \qquad (2)$$

where

$$G = 1 - \left[\frac{r_1 + r_0}{r_1 + r} \right]^4, \qquad (3)$$

Since $r \geqslant r_0$, it follows that

$$0 < \frac{r_1 + r_0}{r_1 + r} \leqslant 1, \qquad (4)$$

whence $G \geqslant 0$ and

$$D > 0 \qquad (5)$$

To obtain an upper bound of D, we write G in the form

$$G = \left[1 + \left(\frac{r_1 + r_0}{r_1 + r} \right)^2 \right] \left[1 + \left(\frac{r_1 + r_0}{r_1 + r} \right) \right] \left[\frac{r - r_0}{r_1 + r} \right],$$

i.e.,

$$G \leqslant 4 \left[\frac{r_1 - r_0 + (r - r_1)}{r_1 + r_0} \right] \qquad (6)$$

But

$$(r - r_1) = \frac{(r^2 - r_1^2)}{r + r_1} \leqslant \frac{r^2 - r_1^2}{r_0 + r_1} = (r_1 - r_0)\, \frac{r^2 - r_1^2}{r_1^2 - r_0^2}$$

Hence,

$$G \lesssim 4 \left(\frac{r_1-r_0}{r_1+r_0}\right)\left[1 - \frac{r_1^2-r^2}{r_1^2-r_0^2}\right] \tag{7}$$

It hence follows for equations (2) and (7) that

$$D < 4N_1\left(\frac{r_1-r_0}{r_1+r_0}\right)\int_{r_0}^{r_1} \frac{r\,dr}{\sqrt{r^2-r_0^2\cos^2E_0}}\left[\left(\frac{r_1^2-r^2}{r_1^2-r_0^2}\right)^4 - \left(\frac{r_1^2-r^2}{r_1^2-r_0^2}\right)^5\right] \tag{8}$$

Making the substitutions

$$x = r^2 - r_0^2\cos^2E_0,\ x_0 = r_0^2-r_0^2\cos^2E_0,\ x_1 = r_1^2-r_0^2\cos^2E_0,$$

inequality (8) may be written in the form

$$D < 4N_1\left(\frac{r_1-r_0}{r_1+r_0}\right)\left[I_4 - I_5\right], \tag{9}$$

where

$$I_n = \frac{1}{2}\int_{x_0}^{x_1} \frac{dx}{\sqrt{x}}\left[\frac{x_1-x}{x_1-x_0}\right]^n \tag{10}$$

Since the total correction Δs_1 may be written in the form

$$\Delta s_1 = N_1\ I_4, \tag{11}$$

it follows that the proportional error in the correction δs_1, $(=D/\Delta s_1)$ must satisfy

$$\delta s_1 < 4\left(\frac{r_1-r_0}{r_1+r_0}\right)\left[1 - \frac{I_5}{I_4}\right] \tag{12}$$

From equations (13) and (14) in Appendix A, we obtain

$$\frac{I_5}{I_4} = \frac{5(1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5)}{6(1 + b_1t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5)} \tag{13}$$

where $a_1 = 5/7$, $a_2 = 5/14$, $a_3 = 5/42$, $a_4 = 1/42$, $a_5 = 1/462$,

$b_1 = 2/3$, $b_2 = 2/7$, $b_3 = 1/14$, $b_4 = 1/126$, $b_5 = 0$.

Since for each s, $a_s > b_s$, it follows that the ratio I_5/I_4 is an increasing function of t. Consequently, $I_5/I_4 \geq 5/6$, and

$$\delta s_1 < 4\left(\frac{r_1 - r_0}{r_1 - r_0}\right)\left(1 - \frac{5}{6}\right),$$

i.e.,

$$\delta s_1 < \frac{2(r_1 - r_0)}{3(r_1 + r_0)} \tag{14}$$

Since $r_1 > r_0$, we finally obtain

$$\delta s_1 < \frac{(r_1 - r_0)}{3r_0} \tag{15}$$

Using the values $r_1 - r_0 = 50\text{km}$, $r_0 = 6000\text{km}$, we conclude that the NWL approximation to the Hopfield formula is accurate to at least one third of one percent.